C3 & C4 Revision Guide

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Chapter 1

Lines

1.1 Introduction

Lines are infinitely thin straight objects in 3D space. These one-dimensional objects can be defined by either a Cartesian equation (such as $y = mx + c$) or in vector form (most common in C3 and C4). The vector form of a line $\vec{l_1}$ in 3D-space consists of the following general equation:

$$\vec{l_1} = \vec{a} + \mu\vec{b} \tag{1.1}$$

Where μ is a real number (i.e. $\mu \in \mathbb{R}$) and $\vec{a}$ and $\vec{b}$ are 3-dimensional vectors (i.e. $\vec{a}, \vec{b} \in \mathbb{R}^3$). In this equation $\vec{a}$ is any point which lies on the line; and $\vec{b}$ is a vector in the direction of the line.

1.2 Parallel Lines

The simplest case is when proving that two lines are parallel. This is simply showing that the two direction vectors are scalar multiples of each other, i.e. for two lines:

$$\vec{l_1} = \vec{a} + \mu\vec{b} \text{ and } \vec{l_2} = \vec{c} + \lambda\vec{d}$$

are parallel if and only if there exists some real value k (also written as $\exists k \in \mathbb{R}$), such that: $\vec{b} = k\vec{d}$.

Example

Show that the lines $\vec{r_1} = \begin{pmatrix} 3 \\ -9 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -10 \\ 0 \end{pmatrix}$ and $\vec{r_2} = \begin{pmatrix} -2 \\ 5 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ are parallel

The two lines are parallel if $\exists k \in \mathbb{R}$ such that:

$$\begin{pmatrix} 5 \\ -10 \\ 0 \end{pmatrix} = k \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

If we set $k = 5$ this is true, therefore the two lines are parallel ($\vec{r_1} \| \vec{r_2}$).

1.3 Intersections

Proving that two lines intersect is a matter of showing that there is a common point on both lines. For example, if we take our two lines from before $\vec{l_1}$ and $\vec{l_2}$, this is equivalent to stating $\exists \lambda, \mu \in \mathbb{R}$ such that:

$$\vec{a} + \mu\vec{b} = \vec{c} + \lambda\vec{d}$$

The way to do this in practise is to write the expression as three simultaneous equations, i.e. for $\vec{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$, $\vec{c} = \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix}$, $\vec{d} = \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix}$, we have the following system of equations:

$$\begin{aligned} a_x + \mu b_x &= c_x + \lambda d_x \\ a_y + \mu b_y &= c_y + \lambda d_y \\ a_z + \mu b_z &= c_z + \lambda d_z \end{aligned}$$

You can solve the first 2 linear equations to find the unique pair (μ, λ) which satisfy them. You then have to show that the equations are consistent by plugging in the values of μ and λ into the third equation and showing that the left-hand side is equivalent to the right-hand side.

Example

Show that the there exists a point P where the lines $\vec{l_1} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$ and $\vec{l_2} = \begin{pmatrix} 9 \\ 7 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$ intersect, and find the co-ordinates of point P.

Firstly we need to make sure that the lines are not parallel, this is easy to see as there is no $k \in \mathbb{R}$ such that $\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = k \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$. Next we need to set up our linear equations for $\vec{l_1} = \vec{l_2}$:

$$\begin{aligned} 2 + \mu &= 9 - 2\lambda \\ 4\mu - 3 &= 7 + \lambda \\ 5 - 2\mu &= 5 - 3\lambda \end{aligned}$$

Solving the first two equations simultaneously, we get $\mu = 3$ and $\lambda = 2$. Putting this into the last equation we get:

$$5 - 2(3) = -1 = 5 - 3(2)$$

Therefore the equations are consistent and the lines intersect. To find the co-ordinates of the point of intersection we simply plug in the values for μ and λ. So:

$$\overrightarrow{OP} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ -1 \end{pmatrix}$$

Therefore we have found the point of intersection is $P(5, 9, -1)$.

1.4 Skew Lines

Proving that two lines are skew is just showing that the lines are both not parallel and have no common point (i.e. $\nexists \mu, \lambda \in \mathbb{R}$ such that $\vec{l_1} = \vec{l_2}$). This can be done in a way similar to showing that two lines have a point of intersection, by setting up the linear simultaneous equations and solving for the first two equations to find the unique pair (μ, λ) which satisfy them, but then showing that these values of μ and λ do not satisfy the third equation and therefore the three equations are inconsistent.

Example

Show that the two lines $\vec{r_1} = (5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}) + s(3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ and $\vec{r_2} = (2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 7\hat{\mathbf{k}}) + t(2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 5\hat{\mathbf{k}})$ are skew.

First we begin by showing that they are not parallel, i.e. $k \in \mathbb{R}$ such that $\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} = k \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix}$, therefore the lines either intersect or are skew. The lines are skew if and only if $s, t \in \mathbb{R}$ such that $\vec{r_1} = \vec{r_2}$, so setting up the system of equations:

$$\begin{aligned} 5 + 3s &= 2 + 2t \\ -2 - 4s &= -2 - t \\ 2s - 2 &= 7 - 5t \end{aligned}$$

Solving the first two equations simultaneously yields: $s = \frac{3}{5}$ and $t = \frac{12}{5}$. Plugging these parameters into the final equation yields:

$$2\left(\frac{3}{5}\right) - 2 = -\frac{4}{5} \neq -5 = 7 - 5\left(\frac{12}{5}\right)$$

Therefore the equations are inconsistent and the lines are skew. **Q.E.D.**

1.5 Angles between lines

They may also ask you to find the angle between two lines, this is a simple matter of using the scalar (or dot) product and the magnitude of the two vectors. As a reminder, the formula for the dot product on two vectors $\vec{u}$ and $\vec{v}$ is:

$$\vec{u} \cdot \vec{v} \equiv ||\vec{u}||||\vec{v}|| \cos\theta = u_x v_x + u_y v_y + u_z v_z \tag{1.2}$$

Therefore you can find the angle between two lines by simply taking the inverse cosine of the dot product of the two direction vectors divided by the magnitude of the two vectors, i.e. for two lines $\vec{l_1} = \vec{a} + \mu\vec{b}$ and $\vec{l_2} = \vec{c} + \lambda\vec{d}$, we can find the angle θ between them:

$$\theta = \cos^{-1}\left(\frac{\vec{b}\cdot\vec{d}}{||\vec{b}||||\vec{d}||}\right) \tag{1.3}$$

Sometimes they ask you to find the *acute* angle ϕ between two lines, so if you find that $\theta > 90°$ then you can easily find $\phi = 180° - \theta$, else $\phi = \theta$.

Example

Find the angle θ between the two lines $\vec{l_1} = \begin{pmatrix} 3 \\ -5 \\ 9 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\vec{l_2} = \begin{pmatrix} 0 \\ 5 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 5 \\ 0 \\ -5 \end{pmatrix}$

Using formula (2) we can easily see that:

$$\theta = \cos^{-1}\left(\frac{\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}\cdot\begin{pmatrix} 5 \\ 0 \\ -5 \end{pmatrix}}{\left\|\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}\right\|\left\|\begin{pmatrix} 5 \\ 0 \\ -5 \end{pmatrix}\right\|}\right) = \cos^{-1}\left(\frac{5+15}{\sqrt{1+4+9}\sqrt{25+25}}\right) = \cos^{-1}\left(\frac{20}{10\sqrt{7}}\right) \approx 40.89°$$

Chapter 2

Differentiation and Integration

2.1 Differentiation

Differentiation is concerned with finding the rate of change of one variable with respect to another (i.e. how the variable changes in response to an infintesimal change in the other variable). So if we have two real variables x and y, then $\frac{dy}{dx}$ is the rate of change of y with respect to x and $\frac{dx}{dy}$ is the rate of change of x with respect to y. The derivative of a function is also written with a $'$ (read "prime"), e.g. $(\cos(x))'$.

2.1.1 Product Rule

The product rule is a rule of differentiation which applies to the product of two functions of the variable with respect to which you are differentiating. That is, for two functions $u(x)$ and $v(x)$, we have:

$$(u(x)v(x)))' = u'(x)v(x) + u(x)v'(x) \tag{2.1}$$

This can be further extended to three or more variables by combining functions, e.g. $u(x)v(x)w(x) = t(x)w(x)$, where $t(x) = u(x)v(x)$. Therefore we get:

$$(u(x)v(x)w(x))' = u'(x)v(x)w(x) + u(x)v'(x)w(x) + u(x)v(x)w'(x) \tag{2.2}$$

It is easy to see how this can be extended to any arbitrary number of functions by taking the sum of the derivative of each of the functions multiplied by the other functions.

Example

Find the derivative with respect to x of the function $y(x) = x^2 \cos x$.

First we recognise that this can be split into the derivative of the product of two functions x^2 and $\cos x$. Therefore we have:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right)\cos x + x^2 \frac{d}{dx}(\cos x)$$

Knowing that $\frac{d}{dx}\left(x^2\right) = 2x$ and that $\frac{d}{dx}(\cos x) = -\sin x$ means that:

$$\frac{dy}{dx} = 2x\cos x - x^2 \sin x$$

2.1.2 Chain Rule

The chain rule is a rule of differentiation which applies to finding the derivative of the composition of two functions. That is for two functions $u(x)$ and $v(x)$, we have:

$$(u(v(x)))' = u'(v(x))v'(x) \tag{2.3}$$

This can be extended to three or more variables in the same way as was done with the product rule; but it is unlikely that you will be asked to do that in the exam.

C3 Exams also make extensive use of the fact that:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \tag{2.4}$$

Example 1

Find the derivative of the function $\sin x^3$ with respect to x

We can apply the chain rule to this by recognising that is in the form $u(v(x))$ where $u(x) = \sin x$ and $v(x) = x^3$, and that $u'(x) = \cos x$ and $v'(x) = 3x^2$. Therefore we have:

$$\frac{d}{dx}\left(\sin x^3\right) = \cos\left(x^3\right) \times 3x^2$$

Example 2 (Exam Question)

Leaking oil is forming a circular patch on the surface of the sea. The area of the patch is increasing at a rate of 250 square meters per hour. Find the rate at which the radius of the patch is increasing at the instant when the area of the patch is 1900 square meters. Give your answer correct to 2 significant figures.

With questions like these it is a good idea to write down the information that you are given in a mathematical form. Firstly, we are given the rate of change of the area of the patch, i.e. $\frac{dA}{dt} = 250$. We are also given the area at the time when we are trying to find the rate of change of the radius: $A = 1900$. Finally we write down what we are trying to find, in this case the rate of change of radius, i.e. $\frac{dr}{dt}$.

Using the chain rule we can write that:

$$\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt}$$

We are given $\frac{dA}{dt} = 250$, and so we must find $\frac{dr}{dA}$. As area is a function of radius it is helpful to use the inverse function theorem, which tells us that $\frac{dr}{dA} = \left(\frac{dA}{dr}\right)^{-1}$. As $A = \pi r^2$, we can then easily find that $\frac{dr}{dA} = \frac{1}{2\pi r}$. We now have everything we need to write the general formula for $\frac{dr}{dt}$: $\frac{dr}{dt} = \frac{250}{2\pi r}$. However, we are asked to find $\frac{dr}{dt}$ at a particular instant, so we need to find the variable r when $A = 1900$.

As $A = \pi r^2$, we can rearrange to get r in terms of A: $r = \sqrt{\frac{A}{\pi}}$, therefore when $A = 1900$ we have $r = \sqrt{\frac{1900}{\pi}} \approx 25.59$. We can then plug this into our general equation for $\frac{dr}{dt}$, giving us:

$$\frac{dr}{dt} = \frac{250}{51.18\pi} \approx 1.62\text{ms}^{-1}$$

2.1.3 Quotient Rule

The quotient rule is a rule of differentiation which applies to finding the derivative of one function divided by another. It can be easily derived from the chain rule and product rule but you will not be asked to do this in the exam. For two functions $u(x)$ and $v(x)$ the quotient rule can be written as:

$$\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x)v(x) - u(x)v'(x)}{\left(v(x)\right)^2} \tag{2.5}$$

This can be found is your green formula booklet so there is no need to remember it, you just need to understand how to apply it.

Example

Prove that the derivative with respect to x of $\tan x$ is $\sec^2 x$.

Firstly we know that $\tan x \equiv \frac{\sin x}{\cos x}$ and that $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$. Therefore, directly applying the quotient rule:

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Q.E.D.

2.2 Integration

Integration can be used for two main purposes: As the opposite operation to differentiation (it is also sometimes called the antiderivative) and in order to find the area under a curve between two points.

2.2.1 Integration by substitution

A common way of integrating less-standard functions is to use a substitution, this often allows for dramatic simplification of the integral. Integration by substitution can be thought of as being used to 'undo' the chain rule. I.e.

$$\int u(v(x))v'(x)\,dx = \int u(t)\,dt \tag{2.6}$$

Or with limits:

$$\int_a^b u(v(x))v'(x)\,dx = \int_{v(a)}^{v(b)} u(x)\,dx \tag{2.7}$$

Example 1

Evaluate the integral $\int x\cos(x^2+5)\,dx$.

We can see that $\frac{d}{dx}\left(x^2+5\right) = 2x$, which suggests that the substitution $u = x^2+5 \implies du = 2x\,dx$ would be a good idea. Using this substitution we have:

$$\int x\cos(x^2+5)\,dx = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + c$$

We then back substitute $u = x^2+5$ to give:

$$\frac{1}{2}\sin u + c = \frac{1}{2}\sin(x^2+5) + c$$

Example 2

Evaluate the definite integral $\int_1^9 \frac{dx}{x(1+\sqrt{x})}$ using the substitution $u = \sqrt{x}$.

Often in questions like this where the substitution may be ambiguous or when it is difficult to find you will be told the substitution to use. In this case we have been told to use $u = \sqrt{x}$ and so we can immediately get that $du = \frac{dx}{2\sqrt{x}}$. In order for this substitution to be useful, we need a factor of $\frac{1}{\sqrt{x}}$ in the integrand. We can get this factor by recognising that $x = \sqrt{x}\times\sqrt{x}$, giving us:

$$\int_1^9 \frac{dx}{x(1+\sqrt{x})} = \int_1^9 \left(\frac{1}{\sqrt{x}(1+\sqrt{x})}\frac{2dx}{2\sqrt{x}}\right) = \int_{x=1}^{x=9} \frac{2du}{u(1+u)}$$

We now need to change the $x = 1$ and $x = 9$ limits into limits with respect to u. As $u = \sqrt{x}$ we have $u = \sqrt{1} = 1$ for the lower limit and $u = \sqrt{9} = 3$ for the upper limit, so:

$$\int_1^9 \frac{dx}{x(1+\sqrt{x})} = \int_1^3 \frac{2du}{u(1+u)}$$

Now in order to evaluate this integral we must use the partial fraction decomposition of $\frac{2}{u(1+u)}$. This turns out to be:

$$\frac{2}{u(1+u)} \equiv \frac{2}{u} - \frac{2}{1+u}$$

Therefore, plugging this into the integral gives:

$$\int_1^9 \frac{dx}{x(1+\sqrt{x})} = 2\int_1^3 \frac{du}{u} - 2\int_1^3 \frac{du}{1+u} = 2\left(\ln(3) + \ln(2) - \ln(4)\right) = \ln\left(\frac{9}{4}\right)$$

2.2.2 Integration by parts

Like Integration by substitution; Integration by parts is the inverse operation of a differentiation rule, in this case the product rule. Integration by parts states:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx \tag{2.8}$$

Or with limits:

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)|_{x=a}^{b} - \int_a^b u'(x)v(x)\,dx \tag{2.9}$$

Example 1

Find the integral with respect to x of $x \ln x$.

In this case the function we cannot integrate is $\ln x$, so we set $u(x) = \ln x$ and $v'(x) = x$, giving us also $u'(x) = \frac{1}{x}$ and $v(x) = \frac{x^2}{2}$. Therefore, applying the formula for the chain rule, we get:

$$\int x \ln x\,dx = \frac{x^2 \ln x}{2} - \frac{1}{2}\int x\,dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$$

In general, in A-level maths exams, if there is a logarithm in the integral then you use that for $u(x)$.

Example 2

Evaluate the definite integral $\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$.

For this integral it is wise to set $u(x) = x$, $u'(x) = 1$, $v'(x) = \sec^2 x$, $v(x) = \int \sec^2 x \, dx = \tan x$. We therefore get:

$$\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx = x \tan x|_{x=0}^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x \, dx$$

We know that $\int \tan x \, dx = \ln|\sec x|$ (this is in the green formula booklet). Therefore:

$$\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx = \frac{\pi}{4} - \frac{1}{2}\ln 2 + c$$

Example 3

Find $\int e^x \sin x \, dx$ using integration by parts.

Here we have two non-polynomial functions e^x and $\sin x$. As e^x is easy to integrate we set $v'(x) = e^x \implies v(x) = e^x$ and therefore $u(x) = \sin x$ and $u'(x) = \cos x$. Using integration by parts we therefore have:

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$$

However, we have the same situation with integrating $e^x \cos x$, so we integrate by parts again with $v'(x) = e^x$, $v(x) = e^x$, $u(x) = \cos x$ and $u'(x) = -\sin x$:

$$\int e^x \sin x \, dx = e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx\right]$$

Now we've ended up with another exponential-trigonometric integral, but this one is the same as before, so if we introduce the variable: $I := \int e^x \sin x \, dx$, we can rewrite the last equation:

$$I = e^x \sin x - e^x \cos x - I \implies 2I = e^x (\sin x - \cos x)$$

And therefore: $I = \int e^x \sin x \, dx = \frac{e^x(\sin x - \cos x)}{2} + c$.

2.3 Differential Equations

Differential equations are equations which contain one or more derivatives within them, $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 5y = 0$, $\frac{dy}{dx} = \frac{2x}{3y}$ and $\frac{dy}{dx} = 3x$ are all examples of differential equations. In C3 and C4 you will only come across two types of differential equations, $\frac{dy}{dx} = f(x)$ to which the solution is simply $\int f(x) \, dx$ and first-order separable differential equations, which are of the form: $\frac{dy}{dx} = f(x)g(y)$.

2.3.1 Separable Differential Equations

As mentioned above, separable differential equations are of the form:

$$\frac{dy}{dx} = f(x) \cdot g(y) \tag{2.10}$$

Where $f(x)$ and $g(y)$ are non-zero functions of x and y respectively. Separable differential equations are solved, as the name suggests, by separating the $f(x)$ and $g(y)$ as follows:

$$\frac{dy}{dx} = f(x) \cdot g(y) \implies \frac{dy}{g(y)} = f(x)\,dx \tag{2.11}$$

We then integrate both sides to give:

$$\int \frac{dy}{g(y)} = \int f(x)\,dx \tag{2.12}$$

Example 1

Find all the solutions to the equation $\frac{dy}{dx} = xy$

This is obviously a separable differential equation of the form: $f(x) = x$ and $g(y) = y$. Therefore, re-arranging, we get:

$$\frac{dy}{y} = x\,dx \implies \int \frac{dy}{y} = \int x\,dx$$

Integrating both sides we get:

$$\ln|y| = \frac{x^2}{2} + c$$

This form of solution where the equation contains both functions of y and x is known as an implicit form. Often it is acceptable to leave the solution in its implicit form, but sometimes they will ask for an explicit form (i.e. $y = f(x)$), this often just involves simple rearrangement and algebra, for instance the explicit solution to the differential equation can easily be found by exponentiating both sides:

$$|y| = e^{\frac{x^2}{2}+c} = e^c e^{\frac{x^2}{2}} = Ce^{\frac{x^2}{2}} \implies y = \pm Ce^{\frac{x^2}{2}}$$

Example 2 (Exam Question)

Given that $\frac{dy}{dx} = \frac{(x-1)\sqrt{y^2+1}}{xy}$ and that $y = \sqrt{e^2 - 2e}$ when $x = e$, find a relationship between x and y.

We can immediately see that this is not a differential equation of the form $\frac{dy}{dx} = f(x)$ as it already has functions of y on the right-hand side. Therefore, as this is a C4 exam it must be a separable differential equation. Therefore we should try to write it in the form $\frac{dy}{dx} = f(x) \cdot g(x)$, we can do this as follows:

$$\frac{dy}{dx} = \frac{(x-1)\sqrt{y^2+1}}{xy} = \frac{(x-1)}{x} \times \frac{\sqrt{y^2+1}}{y}$$

Now we can separate the equation to give:

$$\frac{y}{\sqrt{y^2+1}}\,dy = \frac{(x-1)}{x}\,dx$$

Integrating both sides:

$$\int \frac{y}{\sqrt{y^2+1}}\,dy = \int \left(1 - \frac{1}{x}\right)\,dx$$

As the top of the left-hand integrand fraction is the derivative of the bottom, we can easily integrate it to give $\sqrt{y^2+1}$. So we have:

$$\sqrt{y^2+1} = x - \ln|x| + c \implies y^2+1 = x^2 - 2x\ln|x| + \ln^2|x| + c$$

We can rearrange this further to get an explicit form for y:

$$y = \pm\sqrt{x^2 - 2x\ln|x| + \ln^2|x| - 1} + c$$

We are given a value for y for a particular value of x, which allows us to find the value for c and to determine the sign of the square root expression:

$$x = e \implies y = \sqrt{e^2 - 2e} + c \implies c = 0$$

We can also tell that the sign of the square root is positive and therefore we can get our relation between x and y: $y = \sqrt{x^2 - 2x\ln|x| + \ln^2|x| - 1}$.

Chapter 3

Numerical Methods

3.1 Finding Roots

3.1.1 Finding an interval

Often in C3 exams you are asked to show that the root to some equation $f(x) = 0$ lies within some inverval, i.e. if x_0 is the root of the equation to show that $a \leq x_0 \leq b$ for some given a and b. This can be done by using the fact that if $\operatorname{sgn} f(a) \neq \operatorname{sgn} f(b)$ (where $\operatorname{sgn} x$ means the *signum* or sign of x) then there is at least one root in the interval $a \leq x \leq b$. This fact is a consequence of the Intermediate Value Theorem.

You may also be told that the root lies between two values x and y and then be asked to find the answer to a certain degree of accuracy (e.g. to the nearest 1 decimal place). The way to answer these questions is to divide the interval into smaller intervals finding the interval at each precision and then reducing that interval down into further divisions until you are one decimal place more precise than your answer (i.e. if asked to find the root to the nearest n decimal places, the final divisionw will be $(n + 1)$ decimal places).

Example

It is given that the equation $\tan^2 x - x - 2 = 0$ has a real root between 1.0 and 1.5, find the value of the root x_0 correct to one decimal place.

Firstly, we make a table of values of the value of $f(x) = \tan^2 x - x - 2$ between 1.0 and 1.5:

x	1.0	1.1	1.2	1.3	1.4	1.5
$f(x)$	-0.5745	0.7603	3.4160	9.6751	30.215	195.35

We can see from the table that the root lies somewhere between 1.0 and 1.1. To find which value it is closest to we can simply test $f(1.05)$, if it is greater than 0 then we can state that $x_0 = 1.0$ to one decimal place, else $x_0 = 1.1$ to one decimal place. In this case:

$$\tan^2(1.05) - 3.05 = -0.0109 < 0$$

And thus we can state $x_0 = 1.1$ (1 d.p.).

3.1.2 Iterative Formulae

Iterative formulae are a method of finding roots which relies on ever increasingly accurate approximations to a root α based on a recurrence relation derived from the original equation, of the form $x_{n+1} = f(x_n)$; where for greater values of n, x_{n+1} is closer to α, i.e.:

$$\alpha = \lim_{n \to \infty} f(x_n) \tag{3.1}$$

Often in exams they will guide you through this process by giving you either the recurrence relation or by making it extremely easy to derive and also sometimes giving you the starting value, x_1; they will then ask you to find α to a stated degree of accuracy (n decimal places). This is a simple process of plugging x_1 into the equation to find x_2 and then x_2 into the equation to find x_3 and so on until the first n decimal places are unchanging, this is then your value of α.

Example (Exam Question)

Use the iteration formula $x_{n+1} = \tan^{-1}\sqrt{2 + x_n}$ with a suitable starting value to find the root of $\tan^2 x - x - 2 = 0$ between 1.0 and 1.5 correct to 5 decimal places. You should show the outcome of each step of the process.

Using the result of the previous example we know that the equation $\tan^2 x - x - 2 = 0$ has a root close to 1.05, so it makes sense to use this as our starting value, i.e. $x_1 = 1.05$. We therefore have:

$$\begin{aligned} x_2 &= \tan^{-1}\sqrt{2 + 1.050000} \approx 1.050769 \\ x_3 &= \tan^{-1}\sqrt{2 + 1.050769} \approx 1.050823 \\ x_4 &= \tan^{-1}\sqrt{2 + 1.050823} \approx 1.050827 \\ x_5 &= \tan^{-1}\sqrt{2 + 1.050827} \approx 1.050827 \end{aligned}$$

We can now stop as $x_5 = x_4$ up to the first 6 decimal places. Therefore if α is the root of the equation $\tan^2 x - x - 2 = 0$ between 1.0 and 1.5 then $\alpha = 1.05083$ to 5 decimal places.

3.2 Numerical Integration

3.2.1 Trapezium Rule

As definite integration can be thought of as the area under a curve, i.e. $\int_a^b f(x)\,dx$ is the signed area under the curve $f(x)$ between a and b. Therefore we can approximate these using small strips (either rectangles or trapeziums). In C3 and C4 they expect you to be able to integrate using the trapezium rule, which uses n small, equal-width trapeziums to estimate the area under a curve between a and b. The formula for the trapezium rule is:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left(f(x_0) + 2\left(f(x_1) + \cdots + f(x_{n-1})\right) + f(x_n)\right) \qquad (3.2)$$

Where $x_i = a + \frac{b \times i}{n}$.

Example

Estimate $\int_0^5 \frac{dx}{x+1}$ using the trapezium rule with 5 strips and compare it to the exact answer found by integration.

The easiest bit in the question is finding the exact answer by integration so we will do this first to allow us to see if we're going wrong early in calculations. We know that $\int \frac{dx}{x} = \ln|x|$ so we have:

$$\int_0^5 \frac{dx}{x+1} = [\ln|x+1|]_0^5 = \ln 6 - \ln 1 = \ln 6 \approx 1.7918$$

Now we make a table of values for i, x_i and $f(x_i)$ for $i = \{0, 1, \ldots, 5\}$:

i	0	1	2	3	4	5
x_i	0	1	2	3	4	5
$f(x_i)$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$

We can now directly apply the trapezium rule formula to get:

$$\int_0^5 \frac{dx}{x+1} \approx \frac{1}{2}\left(1 + \frac{1}{6} + 2\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right)\right) = \frac{28}{15} \approx 1.8667$$

This is close to our value of 1.7918 which we achieved through analytic integration, with a relative error of $\frac{1.8667-1.7918}{1.7918} \approx 0.0418$.

3.2.2 Simpson's Rule

Another method of numerical integration is using Simpson's rule, which estimates the area under the curve by approximately fitting parabolas (with 1 parabola per 2 strips). The formula for Simpson's rule with n strips (where n is even) is:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}\left(f(x_0)+f(x_n)+4\sum_{k=0}^{\frac{n}{2}-1} f(x_{2k-1})+2\sum_{k=0}^{\frac{n}{2}} f(x_{2k})\right) \quad (3.3)$$

Where $x_i = a + \frac{b\times i}{n}$.

Example (Exam Question)

The definite integral I is defined by: $I := \int_0^6 2^x\,dx$, use simpson's rule with 6 strips to find an approximate value of I.

Firstly we make a table of values for i, x_i and $f(x_i)$ for $i = \{0, 1, \ldots, 5, 6\}$:

i	0	1	2	3	4	5	6
x_i	0	1	2	3	4	5	6
$f(x_i)$	1	2	4	8	16	32	64

We can now directly apply the formula for Simpson's Rule to get:

$$\int_0^6 2^x\,dx \approx \frac{1}{3}\left(1+64+4\left(2+8+32\right)+2\left(4+16\right)\right) = 91$$

Which is very close to the actual answer of: $\frac{2^6-1}{\ln 2} \approx 90.89$.

Chapter 4

Algebra and Functions

4.1 Functions

A function is simply a construct which takes one set of numbers (the *domain* of the function) and maps it to another set of numbers (the *range* of the function). A function can be thought of as a procedure applied to an input (which is a member of the domain) to produce an output (which is a member of the range). In C3 there are two notations used to define functions, for instance, the function which maps x to x^2 could be written in either of the following ways:

$$f : x \mapsto x^2 \quad \text{or} \quad f(x) = x^2 \tag{4.1}$$

If this function's domain is all real numbers (i.e. $x \in \mathbb{R}$), then the range of the function is the set of all non-negative real numbers ($f(x) \in \mathbb{R} : x \geq 0$). This function is known as a many-to-one function. The identity function $\mathrm{id}_x : x \mapsto x$ is a one-to-one function.

4.1.1 Composition

Composition of a function is the name given to taking the function of another function, the composition of two functions $f(x)$ and $g(x)$ is written in the exam as $fg(x) = f(g(x))$. So for instance, if $f(x) = 2x^2$ and $g(x) = 3x - 5$, then:

$$fg(x) = 2(3x-5)^2 = 2\left(9x^2 - 30x + 25\right) = 18x^2 - 60x + 50$$

4.1.2 Finding the range of a function

A common question in C3 exams is to find the range of a function. When doing this it is important to bear in mind the domain of the function. Finding the range is a three step proceedure: First you check the boundary points of the domain (so if the domain of some function is $a \leq x \leq b$ then find $f(a)$ and $f(b)$, if the domain is $\mathbb{R}$ then find $\lim_{x\to\pm\infty} f(x)$, i.e. the value that the function approaches as its input approaches $\pm\infty$). Then you find the local minima and maxima of the function by differentiating and setting $f'(x) = 0$ and solving for x. You then find the maximum and minimum values that $f(x)$ takes out of all the stationary and boundary points. It may often help to draw a sketch graph to get an intuition of what and where the maxima and minima may be.

Example 1

A function f is defined for all real values of x by:

$$f : x \mapsto k(x^2 + 4x)$$

where k is a positive constant. Find the range of f in terms of k.

Following the 3 step rule above, we first find $\lim_{x\to\infty}(f(x)) = \infty$ and $\lim_{x\to-\infty}(f(x)) = \infty$ because there is a positive co-efficient of x^2 (as k is a positive constant). We then differentiate and set the derivative equal to 0 to find the local extrema of the function:

$$f'(x) = 2kx + 4k \qquad \therefore f'(x) = 0 \implies x = -2$$

We then plug this in to f to get the value of the function: $f(-2) = k((-2)^2 + 4(-2)) = -4k$. As there is only one extreme point and k is a positive constant, this must be the minimum of the function. Therefore the range of the function is $f(x) \geq -4k$.

Example 2

The function $f : x \mapsto 4x^2 - 12x + 8$ is defined for $-3 < x < 6$. Find the range of f.

Again following the 3 step rule, we find the value of the function at the boundary points: $f(-3) = 4(-3)^2 - 12(-3) + 8 = 80$ and $f(6) = 4(6)^2 - 12(6) + 8 = 80$. We then differentiate to find the extrema of the function:

$$f'(x) = 8x - 12 \qquad \therefore f'(x) = 0 \implies x = \frac{3}{2}$$

Again we then plug this into f to get the value of the function: $f\left(\frac{3}{2}\right) = 4\left(\frac{3}{2}\right)^2 - 12\left(\frac{3}{2}\right) + 8 = -1$, this is the minimum for the same reasons as example 1; therefore: $-1 \leq f(x) < 80$.

4.1.3 Inverse Functions

An inverse function often denoted as the original function with a superscript -1, e.g. for a function f, its inverse would be denoted f^{-1}. Inverse functions are important because they obey the following identity:

$$f^{-1}(f(x)) \equiv \mathrm{id}_x(x) \equiv x \qquad \forall x \in \mathbb{D} \tag{4.2}$$

Where $\mathbb{D}$ represents the domain of the function f. Inverse functions also have the property that the domain of the inverse function is equivalent to the range of the original function and the range of the inverse function is equivalent to the domain of the original function. If you are given the graph of a function $f(x)$ and asked to sketch the graph of $f^{-1}(x)$ it is important to note that $f^{-1}(x)$ is a reflection of the graph $y = f(x)$ in the line $y = x$.

Finding an inverse function is done by writing the function f in terms of x and then re-arranging to make x the subject. When x is the subject of the equation, you can then rename $f(x) = x$ and $x = f^{-1}(x)$ to get the inverse function.

Example 1 (Exam Question)

The function f is defined by

$$f : x \mapsto 1 + \sqrt{x} \qquad x \geq 0$$

State the domain and range of the inverse function f^{-1} and find an expression for $f^{-1}(x)$.

The domain of the inverse function is the range of f, so we have $x \geq 1$ as $1 + \sqrt{0} = 1$ and $\lim_{x\to\infty}(1 + \sqrt{x}) = \infty$. The range of the inverse function is simply the domain of the original function so we have $f^{-1}(x) \geq 0$.

Finding an expression for $f^{-1}(x)$ is done by simply re-arranging f to make x the subject:

$$f(x) = 1 + \sqrt{x} \implies f(x) - 1 = \sqrt{x} \implies x = (f(x) - 1)^2$$

We then rename $f(x)$ to x and x to $f^{-1}(x)$ to get:

$$f^{-1}(x) = (x - 1)^2 \qquad x \geq 1$$

Example 2

The function $f : x \mapsto e^{1+\sqrt{x}}$ is defined for $x > 4$. Find the range and domain of $f^{-1}(x)$ and find an expression for $f^{-1}(x)$.

Like before the range and domain are easy to find as they are simply the domain and range respectively of the original function. The range of $f^{-1}(x)$ is $f^{-1}(x) > 4$ and the domain is $x > e^3$. We can find the inverse function in exactly the same way as before, by rearranging $f(x)$ to make x the subject:

$$f(x) = e^{1+\sqrt{x}} \implies 1 + \sqrt{x} = \ln(f(x)) \implies x = (\ln(f(x)) - 1)^2$$

Re-arranging we then get: $f^{-1} : x \mapsto (\ln(x) - 1)^2$.

4.1.4 Modulus Function

A particular special function is the modulus function, also known as the absolute value function, which returns the magnitude of a variable, ignoring its sign. The modulus function $|x|$ is defined formally as follows:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \tag{4.3}$$

For C3 and C4 it is necessary to understand how to solve equations and inequalities involving the modulus function. There are two types of equation that you need to solve: $|f(x)| = g(x)$ and $|f(x)| = |g(x)|$. Which are solved by solving the equations: $f(x) = \pm g(x)$ and $(f(x))^2 = (g(x))^2$ respectively.

Example 1

Find the set of values for x such that $|3x - 5| > 2x - 5$.

Using the definition of the modulus function we have $3x - 5 < 5 - 2x$ and $3x - 5 > 2x - 5$. Solving each of these equations separately we get $x < 2$ and $x > 0$ therefore $0 < x < 2$.

Example 2

Find all values of x which satisfy the equality $|2x - 5| = |x + 2|$. As stated earlier, the key to solving these equations is to square the two expressions to get:

$$|2x - 5|^2 = |x + 2|^2 \implies (2x - 5)^2 = (x + 2)^2$$

Expanding and rearranging we get: $4x^2 - 20x + 25 = x^2 + 4x + 4$, moving all the terms onto the left-hand side and simplifying gives $3x^2 - 24x + 21 = 3(x^2 - 8x + 7) = 3(x - 7)(x - 1) = 0$, therefore $x = \{1, 7\}$ are the solutions to the equation.

Example 3 (Exam Question)

Solve the inequality $|2x + 1| \leq |x - 3|$.

Again we begin by squaring both sides of the inequality:

$$|2x + 1|^2 \leq |x - 3|^2 \implies (2x + 1)^2 \leq (x - 3)^2 \implies 4x^2 + 4x + 1 \leq x^2 - 6x + 9$$

Rearranging and simplifying, we then get $3x^2 + 10x - 8 \leq 0$, factorizing this yields $(3x - 2)(x + 4) \leq 0$, which means that $-4 \leq x \leq \frac{2}{3}$.

4.1.5 Transformations of functions

Also often asked in C3 are questions involving transformations of functions. The following transformations should be remembered for the exam:

Function	Description of Transformation
$f(x)$	The function itself with no change.
$\|f(x)\|$	The function has the same shape and position but all negative values the function takes are reflected in the x-axis.
$f(\|x\|)$	The function is the same for all positive values of x but the graph is then reflected in the y-axis.
$f(ax) \quad a \in \mathbb{R}$	The graph is scaled by scale factor $\frac{1}{a}$ in the x-direction.
$af(x) \quad a \in \mathbb{R}$	The graph is scaled by scale factor a in the y-direction.
$f(x)+b \quad b \in \mathbb{R}$	The graph is translated by $\binom{0}{b}$.
$f(x+b) \quad b \in \mathbb{R}$	The graph is translated by $\binom{-b}{0}$.
$af(x+b) \quad a, b \in \mathbb{R}$	The graph is translated by $\binom{-b}{0}$ and then scaled by scale factor a in the y-direction. These operations can happen in any order.
$f(ax+b) \quad a, b \in \mathbb{R}$	The graph is first translated by $\binom{-b}{0}$ and then scaled by scale factor $\frac{1}{a}$ in the x-direction.

Often if stuck whilst trying to remember one of the transformation functions it can help to imagine the straight line $f(x) = x$ and imagining what happens to it when the functions is applied to it (if you're really stuck then it may help to try a few points on the line, making sure some are negative and some are positive and then drawing the graph).

4.1.6 Understanding e^x and $\ln x$

The exponential function e^x also sometimes written $\exp(x)$ is a special function because it is the only function which it's derivative is equal to itself. That is, $f(x) = e^x$ is the only function to satisfy:

$$\frac{d}{dx}(f(x)) = f(x) \tag{4.4}$$

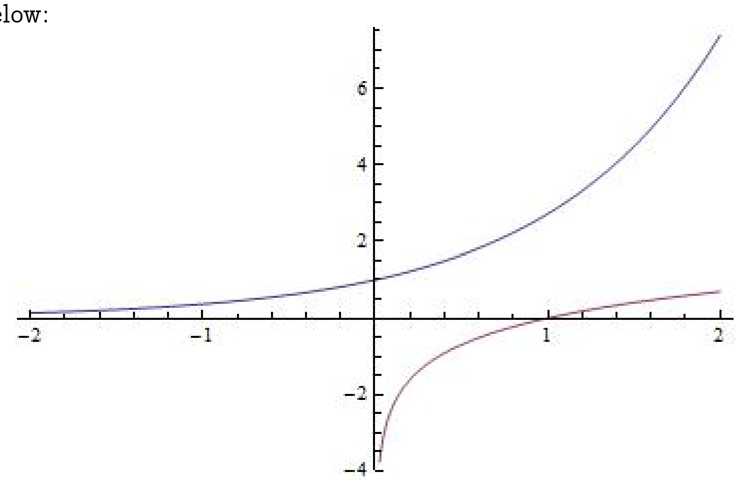
The exponential function has the interesting property that a constant change in the variable x gives the same proportional change (i.e. percentage change) in e^x. The inverse function of e^x is the natural logarithm $\log_e(x)$ usually written $\ln(x)$, such that:

$$\ln(e^x) \equiv e^{\ln(x)} \equiv x \qquad \forall x \in \mathbb{R} \tag{4.5}$$

The derivative of the natural logarithm of x is the reciprocal of x:

$$\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x} \tag{4.6}$$

The combined graph of e^x (shown in blue) and $\ln(x)$ (shown in red) is shown below:



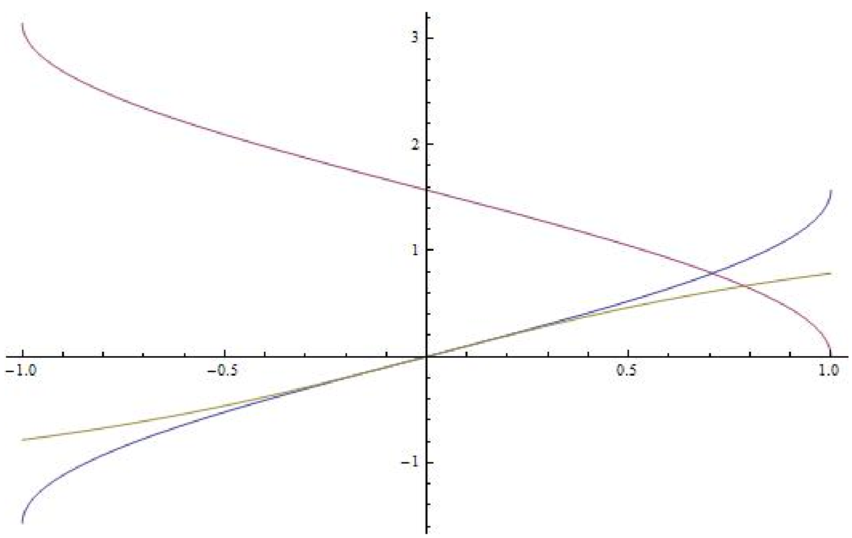
Chapter 5

Trigonometry

5.1 Inverse Trigonometric Functions

For C3 you need to be able to recognise, use and draw the three inverse trigonometric functions: $\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$ (also written as $\arcsin(x)$, $\arccos(x)$ and $\arctan(x)$ respectively). Because of the periodicity of the trigonometric functions, it is nececssary to restrict the domain of the trigonometric functions to $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ for $\sin(x)$ and $\tan(x)$ and $0 \leq x \leq \pi$ for $\cos(x)$. Thus the domain of x for the inverse trigonometric functions are: $-1 \leq x \leq 1$ for $\sin^{-1}(x)$ and $\cos^{-1}(x)$ and $x \in \mathbb{R}$ for $\tan^{-1}(x)$.

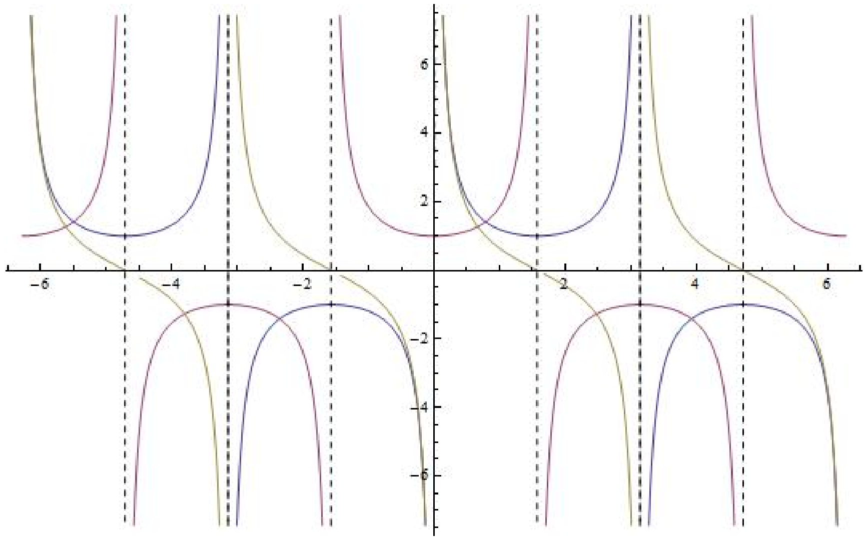
The graphs of $\sin^{-1}(x)$ (shown in blue), $\cos^{-1}(x)$ (shown in red) and $\tan^{-1}(x)$ (shown in gold) for $-1 \leq x \leq 1$ are shown in the graph below:



5.2 Reciprocal Trigonometric Functions

Another important class of trigonometric functions you need to know for C3 and C4 are the reciprocal trigonometric functions: $\csc(x) = \frac{1}{\sin(x)}$, $\sec(x) = \frac{1}{\cos(x)}$ and $\cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}$.

The graphs of $\csc(x)$ (shown in blue), $\sec(x)$ (shown in red) and $\cot(x)$ (shown in gold) are shown in the graph below:



Note that $\csc(x)$ and $\cot(x)$ share the same asymptotes (the bold dashed black lines) and that the asymptotes of $\sec(x)$ are offset from those by $\frac{\pi}{2}$.

5.3 Trigonometric Identities

In C3 and C4 there are several important trigonometric identities you will use. The most important (from C1 and C2) is the equivalence: $\sin^2(x)+\cos^2(x) \equiv 1$; which allows you to derive two others: $1 + \cot^2(x) \equiv \csc^2(x)$ and $\tan^2(x) + 1 \equiv \sec^2(x)$, by dividing by $\sin^2(x)$ and $\cos^2(x)$ respectively; this is not in the green formula booklet so you must be able to remember or derive these as necessary (doing this at the beginning of the exam may be useful).

Below is a table of the main trigonometric identities you will be using (most of which are either the green booklet or can easily be derived from those in the green booklet, ones which aren't in the green booklet are marked by an asterisk $*$):

Identity	Name
$\sin^2(x) + \cos^2(x) \equiv 1 \quad *$	Pythagorean Trigonometric Identity
$1 + \cot^2(x) \equiv \csc^2(x) \quad *$	Pythagorean Trigonometric Identity divided by $\sin^2(x)$
$\tan^2(x) + 1 \equiv \sec^2(x) \quad *$	Pythagorean Trigonometric Identity divided by $\cos^2(x)$
$\sin(\alpha \pm \beta) \equiv \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$	Angle sum and difference identity for sin
$\cos(\alpha \pm \beta) \equiv \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$	Angle sum and difference identity for cos
$\tan(\alpha \pm \beta) \equiv \frac{\tan(\alpha)\pm\tan(\beta)}{1\mp\tan(\alpha)\tan(\beta)}$	Angle sum and difference identity for tan
$\sin(2\theta) \equiv 2\sin(\theta)\cos(\theta) \quad *$	Double angle formula for sin
$\cos(2\theta) \equiv 2\cos^2(\theta) - 1 \equiv 1 - 2\sin^2(\theta) \quad *$	Double angle formula for cos
$\sin^2(\theta) \equiv \frac{1-\cos(2\theta)}{2} \quad *$	Power reduction formula for sin
$\cos^2(\theta) \equiv \frac{1+\cos(2\theta)}{2} \quad *$	Power reduction formula for cos

Example 1 (Exam Question)

Use the identity for $\cos(A + B)$ to prove that:

$$4\cos(\theta + 60^\circ)\cos(\theta + 30^\circ) \equiv \sqrt{3} - 2\sin(2\theta).$$

In questions like these it's best to expand each bit of the right hand side before attempting to do any simplification, so expanding $\cos(\theta + 60^\circ)$ gives:

$$\cos(\theta + 60^\circ) \equiv \cos(\theta)\cos(60^\circ) - \sin(\theta)\sin(60^\circ) \equiv \frac{1}{2}\cos(\theta) - \frac{\sqrt{3}}{2}\sin(\theta)$$

And likewise:

$$\cos(\theta + 30^\circ) \equiv \cos(\theta)\cos(30^\circ) - \sin(\theta)\sin(30^\circ) \equiv \frac{\sqrt{3}}{2}\cos(\theta) - \frac{1}{2}\sin(\theta)$$

Therefore we can expand and simplify the entirety of the left-hand side of the identity to yield:

$$\begin{aligned}
4\cos(\theta + 60^\circ)\cos(\theta + 30^\circ) &\equiv 4\left(\frac{1}{2}\cos(\theta) - \frac{\sqrt{3}}{2}\sin(\theta)\right)\left(\frac{\sqrt{3}}{2}\cos(\theta) - \frac{1}{2}\sin(\theta)\right) \\
&\equiv 4\left(\frac{\sqrt{3}}{4}\left(\cos^2(\theta) + \sin^2(\theta)\right) - \frac{1}{2}\cos(\theta)\sin(\theta)\right) \\
&\equiv \sqrt{3} - 2\cos(\theta)\sin(\theta) \\
&\equiv \sqrt{3} - \sin(2\theta)
\end{aligned}$$

Q.E.D.

Example 2 (Exam Question)

Express the equation $\csc(\theta)(3\cos(2\theta)+7)+11=0$ in the form $a\sin^2(\theta)+b\sin(\theta)+c=0$, where a, b and c are real constants and hence solve the equation for $-180^\circ < \theta < 180^\circ$.

In this case we are given no hint as to what formula to use, so we begin by remembering that $\csc(\theta) \equiv \frac{1}{\sin(\theta)}$. We can then look for other expansions which can be made, such as expanding $\cos(2\theta)$ in terms of sin (as our answer is solely in terms of $\sin(\theta)$).

This allows us to get:

$$\begin{aligned}\csc(\theta)(3\cos(2\theta)+7)+11=0 &\implies \frac{1}{\sin(\theta)}(3-6\sin^2(\theta)+7)+11=0\\ &\implies -6\sin(\theta)+\frac{10}{\sin(\theta)}+11=0\\ &\overset{\times\sin(\theta)}{\implies} -6\sin^2(\theta)+11\sin(\theta)+10=0\end{aligned}$$

Therefore, we have $a=-6, b=11, c=10$, and we can solve the resulting quadratic in $\sin(\theta)$ to get:

$$\sin(\theta) = \left\{\frac{5+\sqrt{85}}{6}, \frac{5-\sqrt{85}}{6}\right\}$$

However, as $-1 \leq \sin(\theta) \leq 1$ and $\frac{5+\sqrt{85}}{6} > 1$, we have the single solution: $\sin(\theta) = \frac{5-\sqrt{85}}{6}$. Solving for θ we get the principle value to be $\sin^{-1}\left(\frac{5-\sqrt{85}}{6}\right) = -44.69^\circ$ (2 d.p.); therefore, by looking at our sine graph over the domain $[-180, 180]$ we can see that the other acceptable value for θ is $-180^\circ + 44.69^\circ = -135.31^\circ$. Therefore our final answer is $\theta = \{-44.69, -135.31\}$.

Example 3 (Exam Question)

Prove the identity

$$\tan(\theta+60^\circ)\tan(\theta-60^\circ) \equiv \frac{\tan^2(\theta)-3}{1-3\tan^2(\theta)}$$

And solve, for $0^\circ < \theta < 180^\circ$, the equation

$$\tan(\theta+60^\circ)\tan(\theta-60^\circ) = 4\sec^2(\theta)-3,$$

giving your answers correct to the nearest 0.1°.

As before, we start by expanding the left-hand side of the equation, using the formulae for $\tan(\alpha \pm \beta)$:

$$\begin{aligned}\tan(\theta+60^\circ)\tan(\theta-60^\circ) &\equiv \left(\frac{\tan(\theta)+\tan(60^\circ)}{1-\tan(\theta)\tan(60^\circ)}\right)\left(\frac{\tan(\theta)-\tan(60^\circ)}{1+\tan(\theta)\tan(60^\circ)}\right)\\ &\equiv \left(\frac{\tan(\theta)+\sqrt{3}}{1-\sqrt{3}\tan(\theta)}\right)\left(\frac{\tan(\theta)-\sqrt{3}}{1+\sqrt{3}\tan(\theta)}\right)\\ &\equiv \frac{\tan^2(\theta)+\sqrt{3}\tan(\theta)-\sqrt{3}\tan(\theta)-3}{1+\sqrt{3}\tan(\theta)-\sqrt{3}\tan(\theta)-3\tan^2(\theta)}\\ &\equiv \frac{\tan^2(\theta)-3}{1-3\tan^2(\theta)}\end{aligned}$$

Q.E.D.

In order to answer the second part, we must use the identity we have just proved, along with the identity: $\sec^2(\theta) \equiv 1+\tan^2(\theta)$ to get:

$$\begin{aligned}\frac{\tan^2(\theta)-3}{1-3\tan^2(\theta)} = 4\left(1+\tan^2(\theta)\right)-3 &\implies \tan^2(\theta)-3 = \left(1+4\tan^2(\theta)\right)\left(1-3\tan^2(\theta)\right)\\ &\implies \tan^2(\theta)-3 = 1+\tan^2(\theta)-12\tan^4(\theta)\\ &\implies 12\tan^4(\theta)=4\\ &\implies \tan^4(\theta)=\frac{4}{12}\end{aligned}$$

We now have a have a quartic in $\tan^4(\theta)$, which we can can solve by taking quartic roots to give: $\tan(\theta) = \pm\sqrt[4]{\frac{4}{12}}$. We can then find the principle values for θ by taking $\tan^{-1}\left(\sqrt[4]{\frac{4}{12}}\right) = 37.23^\circ$ and $\tan^{-1}\left(-\sqrt[4]{\frac{4}{12}}\right) = -37.23^\circ$, however $-37.23^\circ < 0^\circ$ and therefore we add 180° to get 142.77° which is in the specified domain for θ. Therefore our final answer is: $\theta = \{37.2^\circ, 142.8^\circ\}$ (1 d.p.).

Example 4 (Exam Question)

Express $\tan(2\theta)$ in terms of $\tan(\theta)$ and use this identity to deduce the exact value of $\tan(\phi)\cot(2\phi)\tan(4\phi)$ given $\cot(\phi)=4$.

It is given in the formula booklet that $\tan(\alpha+\beta) \equiv \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$, therefore for $\alpha=\beta=\theta$, we have: $\tan(2\theta) \equiv \frac{\tan(\theta)+\tan(\theta)}{1-\tan(\theta)\tan(\theta)} \equiv \frac{2\tan(\theta)}{1-\tan^2(\theta)}$.

We can then expand $\cot(2\phi) \equiv \frac{1}{\tan(2\phi)}$ and $\tan(4\phi) \equiv \tan(2(2\phi))$ using this rule to get:

$$\cot(2\phi) \equiv \frac{1-\tan^2(\phi)}{2\tan(\phi)}$$

And:

$$\tan(4\phi) \equiv \frac{2\tan(2\phi)}{1-\tan^2(2\phi)} \equiv \frac{2\left(\frac{2\tan(\phi)}{1-\tan^2(\phi)}\right)}{1-\left(\frac{2\tan(\phi)}{1-\tan^2(\phi)}\right)^2}$$

We can now write the entirety of $\tan(\phi)\cot(2\phi)\tan(4\phi)$ in terms of $\tan(\phi) = \frac{1}{\cot(\phi)}$, and therefore, plugging in $\tan(\phi) = \frac{1}{4}$ gives:

$$\tan(\phi)\cot(2\phi)\tan(4\phi) = \frac{1}{4} \times \left(\frac{1-\frac{1}{16}}{2\left(\frac{1}{4}\right)}\right) \times \left(\frac{2\left(\frac{2\left(\frac{1}{4}\right)}{1-\frac{1}{16}}\right)}{1-\left(\frac{2\left(\frac{1}{4}\right)}{1-\frac{1}{16}}\right)^2}\right) = \frac{225}{322}$$

Example 5 (Exam Question)

The angles α and β are such that

$$\tan(\alpha) = m+2 \qquad \textbf{and} \qquad \tan(\beta) = m,$$

where m is a constant. Given that $\sec^2(\alpha) - \sec^2(\beta) = 16$, find the value of m and hence find the exact value of $\tan(\alpha+\beta)$.

The key to answering this question is to remember the pythagorean identity $\tan^2(\theta) + 1 = \sec^2(\theta)$; this allows us to find $\sec^2(\alpha)$ and $\sec^2(\beta)$ in terms of m. Applying the identity gives:

$$\sec^2(\alpha) = 1 + \tan^2(\alpha) = 1 + (m+2)^2 = m^2 + 4m + 5$$

And:

$$\sec^2(\beta) = 1 + \tan^2(\beta) = 1 + m^2$$

Which allows us to form the equation in m: $m^2 + 4m + 5 - m^2 - 1 = 16 \implies 4m = 12 \implies m = 3$.

We can find the exact value of $\tan(\alpha+\beta)$ by applying the trigonometric identity $\tan(\alpha+\beta) \equiv \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$, and then plugging in $\tan(\alpha) = m+2 = 5$ and $\tan(\beta) = m = 3$, giving:

$$\tan(\alpha+\beta) = \frac{5+3}{1-5\times 3} = \frac{8}{1-15} = -\frac{8}{14} = -\frac{4}{7}$$

Example 6 (Exam Question)

Given that β is the acute angle such that $\sin(\beta) = \frac{6}{7}$, find the exact value of $\csc(\beta)$ and $\cot^2(\beta)$

This question is a different type of question than the ones above because it relies on the fact that β is an acute angle, and thus we can use the identities of the right angle triangle with $\sin(\beta) = \frac{\text{opp}}{\text{hyp}}$, $\cos(\beta) = \frac{\text{adj}}{\text{hyp}}$ and $\tan(\beta) = \frac{\text{opp}}{\text{adj}}$. From the question we can get that opp $= 6$ and hyp $= 7$, giving us by Pythagoras' theorem: adj $= \sqrt{7^2 - 6^2} = \sqrt{13}$.

Therefore

$$\csc(\beta) = \frac{1}{\sin(\beta)} = \frac{1}{\frac{6}{7}} = \frac{7}{6}$$

And

$$\cot^2(\beta) = \left(\frac{\text{adj}}{\text{hyp}}\right)^2 = \left(\frac{\sqrt{13}}{7}\right)^2 = \frac{13}{49}$$

5.4 Linear Combinations of sin and cos

Due to the properties of the sine and cosine function, it is a known fact that any linear combination of sin and cos (i.e. $\alpha\sin(\theta) + \beta\cos(\theta)$ can be written as a single function in the form $R\sin(\theta+\varphi)$ or $R\cos(\theta+\varphi)$, where $R = \sqrt{\alpha^2+\beta^2}$ is a positive real constant and φ is a real constant found by using the identities for $\sin(\alpha+\beta)$ or $\cos(\alpha+\beta)$ respectively.

If you are asked to express some linear combination $\alpha\sin(\theta) + \beta\cos(\theta)$ in the form $R\sin(\theta+\varphi)$ where R and φ are to be found, the first thing to do is to compute R by taking $\sqrt{\alpha^2+\beta^2}$. Once you have done that you can divide both sides by R to get: $\sin(\theta+\varphi) = \sin(\theta)\cos(\varphi)+\sin(\varphi)\cos(\theta) = \frac{\alpha}{R}\sin(\theta)+\frac{\beta}{R}\cos(\theta)$. From this we can see that $\cos(\varphi) = \frac{\alpha}{R}$ and $\sin(\varphi) = \frac{\beta}{R}$, therefore $\tan(\varphi) = \frac{\beta}{R} \div \frac{\alpha}{R} = \frac{\beta}{\alpha}$, allowing us to get $\varphi = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$.

If, however, you are asked to express the same linear combination in the form $R\cos(\theta+\varphi)$, then you compute R and divide through by it as before but then you apply the trigonometric identity for $\cos(\alpha+\beta)$ to get: $\cos(\theta+\varphi) = \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi) = \frac{\beta}{R}\cos(\theta) + \frac{\alpha}{R}\sin(\theta)$, therefore $\cos(\varphi) = \frac{\beta}{R}$ and $\sin(\varphi) = -\frac{\alpha}{R}$ allowing us to get: $\tan(\varphi) = -\frac{\alpha}{\beta}$ and therefore $\varphi = \tan^{-1}\left(-\frac{\alpha}{\beta}\right)$.

You can either remember the values for φ in the exam, or you can derive them using the methods shown above.

Example 1

Express $3\sin(\theta) + 4\cos(\theta)$ in the form $R\sin(\theta+\psi)$, where R and ψ are to be found.

We first compute the value of $R = \sqrt{3^2+4^2} = \sqrt{25} = 5$, and then divide the whole equation by 5, giving:

$$\frac{3}{5}\sin(\theta) + \frac{4}{5}\cos(\theta) = \sin(\theta+\psi) = \cos(\psi)\sin(\theta) + \sin(\psi)\cos(\theta)$$

We then equate co-efficients of $\sin(\theta)$ to give $\frac{3}{5} = \cos(\psi)$ and $\cos(\theta)$, $\frac{4}{5} = \sin(\psi)$ to give $\tan(\psi) = \frac{3}{4}$ and therefore $\psi = \tan^{-1}\left(\frac{3}{4}\right) = 36.87^\circ$ to 2 decimal places. We can therefore write:

$$3\sin(\theta) + 4\cos(\theta) \equiv 5\sin(\theta + 36.87^\circ)$$

Example 2

The expression $T(\theta)$ is defined for θ in degrees by:

$$T(\theta) = 3\cos(\theta - 60^\circ) + 2\cos(\theta + 60^\circ)$$

Express $T(\theta)$ in the form $A\sin(\theta) + B\cos(\theta)$, giving the exact values of the constants A and B. Hence express $T(\theta)$ in the form $R\sin(\theta + \alpha)$ where $R > 0$ and $0^\circ < \alpha < 90^\circ$.

As we did in previous examples, we first expand the function using the identities $\cos(\alpha + \beta)$ and $\cos(\alpha - \beta)$. This gives us:

$$\begin{aligned}
T(\theta) &= 3\cos(\theta - 60^\circ) + 2\cos(\theta + 60^\circ) \\
&\equiv 3\left(\cos(60^\circ)\cos(\theta) + \sin(60^\circ)\sin(\theta)\right) + 2\left(\cos(60^\circ)\cos(\theta) - \sin(60^\circ)\sin(\theta)\right) \\
&\equiv 3\left(\frac{1}{2}\cos(\theta) + \frac{\sqrt{3}}{2}\sin(\theta)\right) + 2\left(\frac{1}{2}\cos(\theta) - \frac{\sqrt{3}}{2}\sin(\theta)\right) \\
&\equiv \frac{5}{2}\cos(\theta) + \frac{\sqrt{3}}{2}\sin(\theta)
\end{aligned}$$

Therefore $A = \frac{\sqrt{3}}{2}$ and $B = \frac{5}{2}$. We can then compress this linear combination into the form $R\sin(\theta + \alpha)$ as required by taking $R = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \sqrt{7}$. We then divide throughout by $\sqrt{7}$ to get:

$$\frac{\sqrt{3}}{2\sqrt{7}}\sin(\theta) + \frac{5}{2\sqrt{7}}\cos(\theta) = \sin(\theta + \alpha) = \cos(\alpha)\sin(\theta) + \sin(\alpha)\cos(\theta)$$

Which gives $\cos(\alpha) = \frac{\sqrt{3}}{2\sqrt{7}}$ and $\sin(\alpha) = \frac{5}{2\sqrt{7}}$. Therefore $\alpha = \tan^{-1}\left(\frac{5}{\sqrt{3}}\right) = 70.89^\circ$ to 2 decimal places. Thus we can write:

$$T(\theta) = \sqrt{7}\sin(\theta + 70.89^\circ)$$